Kinetic description of avalanching systems

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Avalanching systems are treated analytically using the renormalization group (in the self-organized-criticality regime) or mean-field approximation, respectively. The latter describes the state in terms of the mean number of active and passive sites, without addressing the inhomogeneity in their distribution. This paper goes one step further by proposing a kinetic description of avalanching systems making use of the distribution function for clusters of active sites. We illustrate an application of the kinetic formalism to a model proposed for the description of the avalanching processes in the reconnecting current sheet of the Earth's magnetosphere. A description of avalanching systems is proposed that makes use of the distribution function for clusters of active sites. A general kinetic equation is derived that describes the temporal evolution of the distribution function, in terms of growth and shrinking probabilities. The distribution of clusters is derived for the stationary regime, for a quite general class of avalanching systems or arbitrary dimensionality. The approach, including the probability calculation, is illustrated by an application of the kinetic description to the recently proposed burning model.

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I. INTRODUCTION

Many natural systems, which work in an open configuration, respond to external disturbances showing scale-invariant discrete events [1]. One common feature of these systems is the development of a local threshold instability in an avalanching manner. In the late 1980s, the concept of self-organized criticality (SOC) was proposed by Bak *et al.* [2] for the dynamical-statistical behavior of such systems. SOC has been applied to a variety of systems (see Jensen [3] and references therein for a list of some such systems). Although SOC can exist, strictly speaking, only in the limit of infinitely slow external input, where a complete separation of time scales is achieved [4], it has also been applied to presumably avalanching systems with strong driving. A good example of such systems is space plasma and, in particular, the plasma in the Earth's magnetotail under magnetic substorm conditions [5]. Since SOC is questionable for such strongly driven systems we, in what follows, address to them as to avalanching systems, bearing in mind the avalanchelike propagation of local instabilities. Up to date, the most often used tool for studies of such systems is numerical modeling. The usual analytical approaches proposed so far are the renormalization group methods (see, e.g., Refs. [3,6] and references therein), and the mean-field description (see, e.g., Refs. [3,7] and references therein). The renormalization group methods assume scaling from the very beginning and are applied only in the close vicinity of the stationary (critical) point, that is, in the self-organized criticality regime. The mean-field approach is based on the analysis of the mean number of active, passive, and critical sites. It is not restricted to the criticality range, only including it as the limit of the zero number of active sites. Mean-field approximations predict self-organized criticality in the limit of the zero average number of active sites and, strictly speaking, are applicable only for a system dimension exceeding some critical number, often well above the dimension of real physical systems [8]. Mean-field obtained exponents are often consistent with those found experimentally and numerically for lower dimensions too, but no quantitative explanation is given. On the other hand, deviations from these exponents for real systems are quite usual. The mean-field approach does not take into account the tendency of the active sites to organize in clusters. Indeed, if avalanches of various durations and sizes are present, the distribution of active sites at any moment should be very inhomogeneous. In the present paper we propose a novel approach to the analytical description of avalanching systems that is based on the kinetic equation for the distribution function for active site clusters. We demonstrate the power of the kinetic formalism, applying to the model that was recently proposed as a model of the avalanching reconnection in the current sheet of the Earth's magnetosphere [9].

II. KINETIC EQUATIONS FOR CLUSTERS

The mean-field approach has the obvious drawback of ignoring the fact that active sites have a tendency to appear in clusters. These clusters are, in fact, the instantaneous snapshot of the developing avalanches, so that the size of each cluster is time dependent, $w=w(t)$. However, when considering many coexisting clusters, we may describe their behavior with the help of the distribution function $f(w,t)=dN/dw$, where now the cluster size w and time t are independent variables. The evolution of the single cluster size will be translated into the evolution of the distribution function. The total number of active sites is given by the integral

$$N_a = \int_0^\infty w f(w) dw. \quad (1)$$

We have to introduce also the number of passive sites $N(0)$ (similar to what is done in case of a Bose gas, where the number of particles in the lowest state is macroscopically large). Then $N_a + N(0) = \text{const}$.

Let $P_+(w_1, w_2)$ be the probability of the cluster growth (per unit time), and $P_-(w_1, w_2)$ be the probability of shrinking. Then

$$\frac{\partial f(w)}{\partial t} = \int_0^w P_+(w, w') f(w') dw' + \int_w^\infty P_-(w, w') f(w') dw' - \int_w^\infty P_+(w', w) f(w) dw' - \int_0^w P_-(w', w) f(w) dw' + \gamma(w) N(0) - \frac{f(w)}{\tau(w)}. \quad (2)$$

The term $\gamma(w)N(0)$ in (2) describes the birth of active states due to external driving, while the last term takes into account the finite lifetime of clusters, i.e., the transition to the passive state (Bose-Einstein condensation). If the driving is sufficiently strong and avalanche merging is not negligible, the kinetic equation (2) should be completed with the time-dependent "nonlinear" merging terms,

$$\left(\frac{\partial f(w)}{\partial t}\right)_m = \int P_1(w, w_1, w_2) f(w_1) f(w_2) \times \delta(w - w_1 - w_2) dw_1\, dw_2 - \int P_1(w_1, w, w_2) f(w) f(w_2) \times \delta(w_1 - w - w_2) dw_1\, dw_2. \quad (3)$$

Merging becomes progressively more important when the average fractional density of active sites increases. When this density is not too large (it does not have to be small though, in contrast with the SOC regime), merging will be still relatively weak and can be further studied perturbatively. Strong merging corresponds to the very strong driving, so that the system behavior is, at least partially, forced externally. In the present paper we assume that driving is moderate (not weak and not exceptionally strong) so that merging can be ignored at this stage, deferring treatment of very strongly driven systems to elsewhere. In our case one can expect that there is a wide range (inertial interval) of cluster sizes in which the distribution shape is independent of the external driving and is determined by internal dynamics and/or space dimension.

In general, the distribution function $f(w)$ would depend on the growth and shrinking probabilities. We shall consider here the class of systems where growth and shrinking occur only at the boundaries of clusters. It should be noted that the dynamics inside clusters may induce transitions between active and passive sites, producing, e.g., "punctuated" clusters for the classical sandpile model [2], where an active site becomes passive at the next step. We shall measure the size of such a cluster, including the passive (receiving) sites as well, so that the internal dynamics does not affect the cluster size. The situation may be more complicated when clusters are developed fractals, with tunnels appearing and crossing the cluster [1]. Such systems would probably require special treatment. We restrict ourselves here with the clusters that grow or shrink at their boundaries. Space and laboratory plasma systems [10] seem to belong to this class.

In this case the probabilities are nonzero only for $|w' - w| = \Delta \ll w$, so that (2) can be written as

$$\frac{\partial f}{\partial t} = -\tilde{P}_-(w)\sigma(w)f(w) - \tilde{P}_+(w)\sigma(w)f(w) + \tilde{P}_-(w+\Delta)\sigma(w+\Delta)f(w+\Delta) + \tilde{P}_+(w-\Delta)\sigma(w-\Delta)f(w-\Delta), \quad (4)$$

where $\sigma(w)$ is the density of states. This approximation is not valid for small w, where the cluster kinetics should be strongly affected directly by driving. We seek an approximate description of the cluster kinetics in the range where it is determined, but the internal features of the system rather than by external influence. It is obvious that if a large size strong driving is applied, the reaction of the system would be a forced reaction and not self-organized in any way.

The approximation may be not accurate for largest clusters either, since the possible fractality [1] of clusters may result in the breakdown of independence of probabilities at neighboring active boundary sites. Indeed, all numerical simulations [3] show distortions for very small and very large w. Thus, the physical sense of our approximation is that we are working in the *inertial interval* far from both limits. According to existing analyses, such an interval exists almost always.

For one-dimensional clusters $\sigma(w) = 1$ or $\sigma(w) = 2$ (the latter holds for growth in both directions). This allows an immediate n-dimensional generalization. Let w be a linear measure of a cluster (effective *radius*), and let D be the cluster volume. The density of states σ is then the cluster surface area. In general, $D \propto w^\mu$, $\sigma \propto w^\nu$, $n \geq \mu > \nu \geq n-1$, where μ and ν are fractal dimensions of the cluster volume and boundary, respectively. Taylor expanding (4), we arrive at the following differential equation:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial w}(\alpha\sigma f) + \frac{\partial^2}{\partial w^2}(\beta\sigma f), \quad (5)$$

where $\alpha = \Delta(\tilde{P}_- - \tilde{P}_+)$ and $\beta = (\Delta^2/2)(\tilde{P}_- + \tilde{P}_+)$. The stationary solution,

$$f \propto (1/\beta\sigma)\exp\left[-\int (\alpha/\beta) dw\right], \quad (6)$$

exists only if $\alpha > 0$. In general, α and β can depend on w. Both describe the local growth (shrinking) per site at the cluster surface. Their dependence on the cluster size would mean essentially that the growth and shrinking probabilities as well as variation of the affected neighbor zone at some site depend on what happens at other sites. While, in principle, this cannot be excluded (waves could transfer information across the cluster or long-range forces are involved [1]),

many avalanche systems seem to be governed by local dynamics, so that it is natural to consider (at least at this stage) the case of probabilities independent of w (see, however, the comment in Sec. III). One finds

$$f \propto \sigma^{-1} \exp(-w/w_c) = w^{-\nu} \exp(-w/w_c), \tag{7}$$

with w_c=const. The obtained $f=dN/dw$ describes the distribution of linear sizes (effective radii). For the distribution of the cluster volumes, one has

$$\frac{dN}{dD} = \frac{dw}{dD} \cdot \frac{dN}{dw} \propto D^{(1-\mu-\nu)/\mu} \exp(-AD^{1/\mu}). \tag{8}$$

In the mean-field limit $n \gg 1$ [7], one has $(dN/dD) \propto D^{-2}$.

The derived expressions assume isotropy. If the system is anisotropic and/or a preferential shape of cluster exists, e.g. clusters are elongated [1], the above treatment may have to be modified by considering vector **w** describing linear sizes along principal axes. These modifications are of a technical character and do not change substantially the basic equations and conclusions. Yet, they require a more lengthy analysis and cannot be presented in a letter. We will provide this analysis elsewhere.

III. BURNING MODEL

The above theory can be illustrated on the simple "burning" model [9] described below. In this model each site is characterized by its temperature, $T(x)$. The external driving is random heating of the sites. The amount q of heat per unit time is going to a site with probability p, so that the average heat transfer from outside (in driving) is pq. The temperature of a passive site (the one that is not burning) changes according to

$$\frac{dT}{dt} = qp[1 - \eta(t)], \tag{9}$$

where $\eta(t)$ is a random number, $|\eta| \leq 1$, so that $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$. Once $T > T_c$, where T_c is some critical temperature, the site becomes active. An active site burns and produces heat at the rate $J=\mu T$, $\mu < 1$. During the burning stage the temperature decreases (unless driving is strong enough to force permanent burning). When the temperature drops below some value T_l such that $T < T_l < T_c$, the burning ceases and the site becomes passive again. Part of the heat release is lost (radiated away), while the other part, $2aJ$, is transferred (isotropically) to the closest neighbors. Summarizing the above, the heat release can be written approximately as

$$J = \mu T \theta(T_c - T)\theta(-dT/dt)\theta(T - T_l) + \mu T \theta(T - T_c), \tag{10}$$

where $\theta(x)$ is the step function. The term $\theta(-dT/dt)$ is an approximate manifestation of the history-dependent (hysteresis) burning for $T_c > T > T_l$ (burning now if it was burning at the previous moment/step and not burning otherwise). This expression is not quite correct for the temperature of a site that does not have to change monotonically when an avalanche develops. We leave the more detailed discussion of this for another paper, especially devoted to this model. For the purposes of the present discussion, such details are irrelevant, and we consider (10) as a sufficiently precise description of the burning process. If an active site would be left alone, its temperature would decrease as $T = T(0)\exp(-\mu t)$. Here the quantity $\tau \approx (1/\mu)\ln(T_c/T_l)$ has the meaning of the lifetime of an active site if it were not affected either by other sites or external driving. Let Δ_t be the time step and Δ_l the site size. The amount of heat a site x receives is given by

$$\frac{dT(x)}{dt} = qp[1 - \eta(t)] + a[J(x+\Delta_l) + J(x-\Delta_l)] - J(x), \tag{11}$$

which we write in the following form:

$$\frac{dT(x)}{dt} = qp[1 - \eta(t)] + (2a-1)J + \frac{a\Delta_l^2}{2}\frac{\partial^2 J}{\partial x^2}. \tag{12}$$

Integrating (11) over a cluster of the size w, one gets

$$\frac{d}{dt}\int T\,dx = qpw + (2a-1)\int J\,dx - J_b, \tag{13}$$

where we averaged over time the random fluctuations of the input η. The meaning of the terms on the right-hand side is quite obvious: the first term is the energy input due to external driving, the second term is the radiation losses, and the last term is the heat flux at the cluster boundaries.

The probability of growth should be proportional to the heat flux from the active site at the cluster boundary to the neighboring passive sites. This probability should depend on the temperature of the passive sites. In the stationary regime the time-average growth probability would be determined by the average temperature T_p of passive sites. Thus, growth is essentially independent of the cluster size. Respectively, the shrinking probability depends on the state of the boundary site and is not particularly sensitive to the cluster size either. In this case the parameters α and β are constant, and one expects that the cluster distribution is an exponential, $f \propto \exp(-w/w_0)$. However, if the heat transfer in the active area is suppressed (active sites do not easily accept heat from active neighbors) spreading from the central regions with the constant speed up to the cluster boundaries, one estimates that $P_+ \propto 1/w$, while $P_- \approx$const. In this case $\alpha = \Delta(a_1 - a_2/w)$, and $\beta = (a_1 + a_2/w)\Delta^2/2$, and $f \propto (w + w_0)^{\lambda}\exp(-w/w_c)$, where w_0, w_c, and λ are constants. In the range $w_0 \ll w \ll w_c$ (if this such range does exist at all) a power-law distribution should be observed. In the opposite case, when the heat is transferred immediately from the inside to the cluster boundaries, $P_+ \propto w$ and $P_- \approx$const, no stationary state can exist, since $\alpha < 0$ for sufficiently large w. Such systems are unstable and are disrupted into avalanches that will cover the entire system.

IV. CONCLUSIONS

We proposed a kinetic approach to the description of avalanching systems, defining a distribution function $f(w,t)$ for

the clusters of active sites. In this way we derived a kinetic equation for the temporal evolution of $f(w,t)$ and analyzed its steady-state limit in the inertial range, sufficiently far from the smallest scales where driving explicitly shows up, and sufficiently far from the larges scales where fractality and merging become progressively more important. The stationary distribution function $f(w)$ depends, in general, on the probability of the microprocesses, resulting in cluster growth and shrinking, that is, the processes governing the evolution of avalanches. In the case of locally induced growth at the boundaries, the shape of the distribution is determined by the dimension of the system (or fractal dimensions of clusters if they are not compact). There is no sensitivity to the input details. The obtained universal shape of the distributions is not limited to the weak driving regime or to the system dimension above some critical value, and can be used for direct and easy comparison with experiments and numerical modeling. The total average driving should affect the state of the system, as we have shown in a particular model. The estimates given in the present model represent just the first step toward a more elaborated kinetic model of the dynamics of avalanches. We remark that our analytical predictions have been checked by one-dimensional (1D) and 2D burning model simulations to be reported elsewhere.

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